SEMIGROUP ACTIONS ON SETS

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Abstract. In this paper we discuss constructions of several categories of sets with semigroup actions, by introducing definitions of actions based on bi-actions. We discuss equivalences between some of these categories and discuss the advantages of new constructions over the usual ones already existing in the literature. These categories possess two idempotent endofunctors, which can invert a given action. By using these endofunctors we show that these categories admit structures of a homotopical category. We finally construct the Burnside ring of a monoid by using homotopical structure of these categories and show that when the monoid is commutative then its Burnside ring is equivalent to Burnside ring of its Gröthendieck group.

1. Introduction

Actions of semigroups appear quite often as mathematical models of progressive processes. For example in computer science automata or so called state machines can be defined using semigroup actions. In physics a dynamical system can be seen as a semigroup action. An important problem in the theory of semigroup actions is reversibility of actions. Reversible actions are particularly important when one considers applications. For example, in [8] Landauer establishes the relation of reversibility of computation with energy consumption. Reversibility is also a fundamental issue in the theory of quantum state machines, since a quantum automaton has to be reversible. In dynamical systems the periodic attractors can be considered as reversible parts of the dynamical system.

A monoid having inverses is a group. In the theory of group actions, when a group G is given, one often considers either left actions of G, or right actions of G, or if another group H is given, one considers (G, H)-bisets, i.e. biactions of G and H so that G acts from left and H acts from right. The categories of these actions are also well studied in the literature, see e.g. [3]. The same distinction is also present in semigroup and monoid actions. For a given semigroup or a monoid I, we define actions by fusing previous ideas in a suitable manner, so that we no longer need to call them left, right or biactions, and we call them just "actions".

We construct the category of actions in Section 3, see Lemma 3.1, and we denote the category of all actions by $\mathrm{ACT}(I)$. For groups this category will be equivalent to the one defined in the usual way, so that when I is a group the category of left I-sets (which is equivalent to right I-sets) is equivalent to $\mathrm{ACT}(I)$. In the theory of group actions when a left group action on a set is given, one can define a right action on the same set given by acting with the inverses of elements in the group; which is often called inverse action of the given left action. A similar construction exits for right actions as well. Due to lack of inverses these "inverse

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action" constructions are not possible for actions of semigroups or monoids. On the other hand a generalizations are still possible for semigroup actions on sets by considering our definition of actions. One of the objects of this paper is to define inverse actions so that they generalize the ones for groups. For a semigroup action on a set, these constructions are called "inverting from left to right" and "inverting from right to left", see Section 4.1. Although the inverse actions have to be defined on different sets, it agrees with the above construction up to isomorphism when we consider group actions on sets, see Theorem 4.2. These constructions define two endofunctors on $\mathrm{ACT}(I)$, which will be called the inverting functors, $\mathrm{Inv}^{\mathrm{r}}_{\mathrm{l}}$ and $\mathrm{Inv}^{\mathrm{l}}_{\mathrm{r}}$, which are idempotent, see Lemma 4.3. Composition of these functors will not be idempotent in general but when we restrict our attention to finite I-sets it will be idempotent.

There are several other major advantages of these definitions of actions. For a semigroup I, the category $\mathrm{ACT}(I)$ has an equivalent subcategory denoted by $\overline{\mathrm{ACT}}(I)$, whose objects are actions which are "invertible on one side" with equivariant maps between them, see Section 3. This subcategory $\overline{\mathrm{ACT}}(I)$ possesses a homotopical category structure in the sense of [7], see Section 5. We denote the full-subcategory of $\overline{\mathrm{ACT}}(I)$ of finite I-sets by $\overline{\mathrm{act}}(I)$. In the case when I is a monoid the homotopical category $\overline{\mathrm{act}}(I)$ will admit 3-arrow calculus, so that from 27.5 of [7] it will be saturated. As a result we are able to define Burnside ring of a monoid I, which is another main objective of this note. We denote the Burnside ring of a monoid I by $\mathrm{A}(I)$. When I is a group this definition agrees with the definition of Burnside ring existing in the literature, see [11], which is a very important construction in group theory. If I is a commutative monoid and $K(I)$ is its Gröthendieck group, then we have proved $\mathrm{A}(I)$ is equal to $\mathrm{A}(K(I))$, see Theorem 6.1. In particular we show $\mathrm{A}(\mathbb{N}) = \mathrm{A}(\mathbb{Z})$, so that we can add one more arrow which would be an isomorphism in the main diagram in [5] page 3.

We also recover the idea of the attractors for a state machine in analogy with the attractors in the field of dynamical systems, as the inverse actions of a given monoid action. When the monoid is taken as $\mathbb{N}$, this will correspond to the standard definitions. The periodic attractors will be the generators of the Burnside ring. The Burnside ring of a free monoid on an alphabet is an invariant of the types of machines can be build, so that it would be very useful in Automata theory.

2. Actions of semigroups and monoids on sets

Given sets A and B, we denote the set of functions from A to B by $[A, B]$ and we denote the set of endofunctions on A by $\mathrm{End}(A)$. One can define two distinct monoid structures on the set $\mathrm{End}(A) = [A, A]$, where the identity on A is the identity of the monoid. In the first one we choose the monoid operation on $\mathrm{End}(A)$ as the composition of endofunctions when endofunctions are applied on A from right. Then we denote this monoid by $\mathrm{End}_r(A)$ and we write fg for the composition of f and g in $\mathrm{End}_r(A)$, which we mean f is applied first then g. In other words if f and g are in $\mathrm{End}_r(A)$ and a is in A then

$$(a)(fg) = ((a)f)g.$$

Similarly, for the second one we write $\mathrm{End}_l(A)$ for the monoid obtained by taking the monoid operation on $\mathrm{End}(A)$ as the composition of endofunctions when endofunctions are applied on A from left. In this case we write $f \circ g$ for the composition

of f and g in $\mathrm{End}_l(A)$. In other words if f and g are in $\mathrm{End}_l(A)$ and a is in A then

$$(f \circ g)(a) = f(g(a)).$$

We can also consider the endofunction sets $\mathrm{End}_r(A)$ and $\mathrm{End}_l(A)$ with the underlying semigroup structure.

2.1. **Actions on sets and function sets.** Let I be a semigroup (resp. a monoid). All through this section we denote the operation in I by $\otimes$. Normally one defines an action of I on a set A as a function $A \times I \to A$ which is compatible with the semigroup operation; or alternatively, it can be defined as a semigroup (resp. a monoid) homomorphism from I to $\mathrm{End}_r(A)$ and call it a right action of I on A. One can also define an action of a semigroup I on a set A as a semigroup (resp. a monoid) homomorphism from I to $\mathrm{End}_l(A)$ and call it a left action of I on A. However, in this paper we consider an action of a semigroup (resp. a monoid) on a set as a biaction. More precisely we have the following definition:

Definition 2.1. Suppose that I is a semigroup (resp. monoid) and A is a set. An action α of I on A is a pair (α_l, α_r) such that $\alpha_l : I \to \mathrm{End}_l(A)$ and $\alpha_r : I \to \mathrm{End}_r(A)$ are semigroup homomorphisms (resp. monoid homomorphism) and α_l commutes with α_r so that for all i, j in I and a in A we have

$$(\alpha_l(i)(a))\alpha_r(j) = \alpha_l(i)((a)\alpha_r(j)).$$

Instead of saying α is an action of I on A, we could also say α is a I-action on A or say (A, α) is a I-set or just say A is a I-set.

Suppose that we have I-actions $\alpha = (\alpha_l, \alpha_r)$ on A and $\beta = (\beta_l, \beta_r)$ on B. There is an induced I-action

$$[\alpha, \beta] = ([\alpha, \beta]_l, [\alpha, \beta]_r)$$

on $[A, B]$ such that for f in $[A, B]$ and i in I the function $[\alpha, \beta]_l(i)(f)$ is the composition

$$A \xrightarrow{\alpha_r(i)} A \xrightarrow{f} B \xrightarrow{\beta_l(i)} B$$

and $(f)[\alpha, \beta]_r(i)$ is the composition

$$A \xrightarrow{\alpha_l(i)} A \xrightarrow{f} B \xrightarrow{\beta_r(i)} B.$$

2.2. **Equivariant functions and fixed point sets.** We first are going to define centralizers of semigroup and monoid actions. Let (A, α) be a I-set where $\alpha = (\alpha_l, \alpha_r)$. Then $C_A(I)$ the centralizer of I in A with the action α is defined as

$$C_A(I) = \{a \in A : \forall i \in I, \alpha_r(i)(a) = (a)\alpha_l(i)\}.$$

Suppose that we have I-actions $\alpha = (\alpha_l, \alpha_r)$ on A and $\beta = (\beta_l, \beta_r)$ on B. Considering the I-action $[\alpha, \beta]$ on $[A, B]$ we define $Map_I(A, B)$ namely the set of I-equivariant functions from A to B as of I in $[A, B]$ with the induced action $[\alpha, \beta]$, i.e.

$$\mathrm{Map}_I(A, B) = C_{[A,B]}(I).$$

Hence a function is called a I-equivariant function from A to B if it is in $\mathrm{Map}_I(A, B)$, so that a function $f : A \to B$ is a I-equivariant function if and only if we have the identity

$$(f(\alpha_l(i)(a)))\beta_r(i) = \beta_l(i)(f((a)\alpha_r(i)))$$

for all i in I and a in A.

Here we list some of the properties of equivariant functions similar to the classical case. Let (A, α), (B, β), (C, γ), and (D, δ) be four I-sets. Assume $f : A \to B$ and $h : C \to D$ be two functions. The functions f and h induces a function $[B, C] \to [A, D]$ which sends $g : B \to C$ to $h \circ g \circ f$. The following result shows that compositions by equivariant functions induces an equivariant function between function sets.

Proposition 2.2. *If $f : A \to B$ and $h : C \to D$ are two I-equivariant functions then the induced function $[B, C] \to [A, D]$ by f and h is I-equivariant.*

Proof. Since f and h are I-equivariant we have

$$(h(\gamma_l(i)(g((f(\alpha_l(i)(a)))\beta_r(i)))))\delta_r(i) = \delta_l(i)(h((g(\beta_l(i)(f((a)\alpha_r(i)))))\gamma_r(i)))$$

for all a in A, i in I and g in $[B, C]$. Hence we have

$$(h \circ (([\beta, \gamma]_l(i)(g)) \circ f))[\alpha, \delta]_r(i) = [\alpha, \delta]_l(i)(h \circ (((g)[\beta, \gamma]_r(i)) \circ f))$$

for all i in I and g in $[B, C]$. This means the induced function from $[B, C]$ to $[A, D]$ is I-equivariant. $\square$

Let A be a I-set. Then we define $\mathrm{Fix}_I(A)$ namely the set of fix points of I on A as

$$\mathrm{Fix}_I(A) = \mathrm{Map}_I(*, A)$$

where $*$ denotes a set with one element and the trivial I-action on it.

Proposition 2.3. *Let I be a semigroup or a monoid, and A, B be two I-sets. Then we have a bijection*

$$\mathrm{Map}_I(A, B) \cong \mathrm{Fix}_I([A, B]).$$

Proof. More generally for an I-set X we have a bijection from $C_I(X)$ to $C_I([*, X])$ sending z in $C_I(X)$ to the function from $*$ to X which sends the unique point in $*$ to z. $\square$

Given a function $f : A \to [B, C]$ we define $\bar{f} : A \times B \to C$ by $\bar{f}(a, b) = f(a)(b)$ for all a in A and b in B.

Lemma 2.4. *Let A, B and C be three I-sets with I-actions α, β and γ respectively. Then the function*

$$\mathrm{Map}_I(A, [B, C]) \to \mathrm{Map}_I(A \times B, C)$$

defined by $f \mapsto \bar{f}$ is a bijection.

Proof. We only need to show that $f : A \to [B, C]$ is a I-equivariant function if and only if $\bar{f} : A \times B \to C$ is a I-equivariant function. We know that the statement $f : A \to [B, C]$ is a I-equivariant function means

$$(f)[\alpha, [\beta, \gamma]]_r(i) = [\alpha, [\beta, \gamma]]_l(i)(f)$$

for all i in I. In other words it means

$$(f(\alpha_l(i)(a))(\beta_l(i)(b)))\gamma_r(i) = \gamma_l(i)(f((a)\alpha_r(i))((b)\beta_r(i)))$$

for all a in A, b in B and i in I. Hence it is equivalent to

$$(\bar{f}(\alpha_l(i)(a), \beta_l(i)(b)))\gamma_r(i) = \gamma_l(i)(\bar{f}((a)\alpha_r(i), (b)\beta_r(i)))$$

for all a in A, b in B and i in I. Therefore the statement $f : A \to [B, C]$ is a I-equivariant function is equivalent to

$$(\bar{f})[\alpha \times \beta, \gamma]_r(i) = [\alpha \times \beta, \gamma]_l(i)(\bar{f})$$

which means $\bar{f} : A \times B \to C$ is a I-equivariant function. □

3. Categories of I-sets

In this section, for a semigroup or a monoid I, we will define several categories whose objects are sets with an action of I defined in the usual sense or in the sense of previous section. In each case the morphisms of the category will be I-equivariant functions defined according to the case being considered. In order to define objects of these categories we will first discuss semi-invertible actions and actions which are invertible on one side. Second we will show that the composition of two I-equivariant functions is an I-equivariant function under certain conditions. Finally we will give the definitions of categories of certain I-sets and discuss equivalences between some of these categories when I is a group.

3.1. **Semi-invertible actions and actions invertible on one side.** Let (A, α) be an I-set. First note that if $\alpha_l(i)$ is an automorphism of A then for all a in A we have the equality

$$\alpha_l(i)^{-1}((a)\alpha_r(j)) = (\alpha_l(i)^{-1}(a))\alpha_r(j).$$

and similarly in the case when $\alpha_r(i)$ is an automorphism of A then we have

$$\alpha_l(i)((a)\alpha_r(j)^{-1}) = (\alpha_l(i)(a))\alpha_r(j)^{-1}.$$

We say (A, α) is "semi-invertible" if either $\alpha_l(i)$ or $\alpha_r(i)$ is an automorphism of A for all i in I and we say (A, α) is "invertible on one side" if either $\alpha_l(i)$ is an automorphism of A for all i in I or $\alpha_r(i)$ is an automorphism of A for all i in I. Note that if an action is invertible on one side then it is semi-invertible. Hence the results about semi-invertible actions in this section are also true for actions that are invertible on one side.

3.2. **Compositions of equivariant functions.** Compositions of equivariant functions may not be equivariant unless we have a semi-invertibility assumption in the following sense. Let S be a set and $(B_s, \beta(s))$ be a semi-invertible I-set for s in S. Define B as the product $\Pi_{s\in S}B_s$ with the I-action given by $\beta(s)$ on the s^{th} component. Assume (A, α) and (C, γ) are I-sets and $f : A \to B$, $g : B \to C$ are I-equivariant functions. Then we have the following result

Lemma 3.1. *$g \circ f$ is I-equivariant.*

Proof. We want to show

$$\gamma_l(i)((g \circ f)((a)\alpha_r(i))) = ((g \circ f)(\alpha_l(i)(a)))\gamma_r(i)$$

for any a in A and i in I. Let us denote the left-hand side of above equality by LHS and the right-hand side by RHS. Let f_s denote the s^{th} component of f. Given any s in S and i in I, since $(B_s, \beta(s))$ is semi-invertible, there exists $x(s, i)$ in $\{l, r\}$

such that $\beta(s)_{x(s,i)}(i)$ is an automorphism of B_s. Since $\beta(s)_{x(s,i)}(i)^{-1}\circ\beta(s)_{x(s,i)}(i)$ is identity we have

$$\begin{aligned}\text{LHS} &= \gamma_l(i)(g(f((a)\alpha_r(i)))\\ &= \gamma_l(i)(g((f_s((a)\alpha_r(i)))_{s\in S}))\\ &= \gamma_l(i)(g((E(a,i)_s)_{s\in S}))\end{aligned}$$

where

$$E(a,i)_s = \begin{cases} (\beta(s)_l(i)^{-1}\circ\beta(s)_l(i))(f_s((a)\alpha_r(i))) & \text{if } x(s,i)=l\\ (f_s((a)\alpha_r(i)))(\beta(s)_r(i)^{-1}\beta(s)_r(i)) & \text{if } x(s,i)=r\end{cases}$$

We have

$$\text{LHS} = \gamma_l(i)(g((F(a,i)_s)_{s\in S}))$$

if $F(a,i)_s$ is defined as follows:

$$F(a,i)_s = \begin{cases} \beta(s)_l(i)^{-1}(\beta(s)_l(i)(f_s((a)\alpha_r(i)))) & \text{if } x(s,i)=l\\ ((f_s((a)\alpha_r(i)))\beta(s)_r(i)^{-1})\beta(s)_r(i) & \text{if } x(s,i)=r\end{cases}$$

Since f is I-equivariant means f_s is I-equivaraint for all s in S we have

$$\text{LHS} = \gamma_l(i)(g((G(a,i)_s)_{s\in S}))$$

where

$$G(a,i)_s = \begin{cases} \beta(s)_l(i)^{-1}((f_s(\alpha_l(i)(a)))\beta(s)_r(i)) & \text{if } x(s,i)=l\\ ((f_s((a)\alpha_r(i)))\beta(s)_r(i)^{-1})\beta(s)_r(i) & \text{if } x(s,i)=r\end{cases}$$

By the above equality

$$\text{LHS} = (g(\beta(s)_l(i)((H(a,i)_s))_{s\in S}))\gamma_r(i)$$

with

$$H(a,i)_s = \begin{cases} \beta(s)_l(i)^{-1}(f_s(\alpha_l(i)(a))) & \text{if } x(s,i)=r\\ (f((a)\alpha_r(i)))\beta(s)_r(i)^{-1} & \text{if } x(s,i)=r\end{cases}$$

Since g is I-equivariant

$$\begin{aligned}\text{LHS} &= (g((J(a,i)_s)_{s\in S}))\gamma_r(i)\\ &= (g((K(a,i)_s)_{s\in S}))\gamma_r(i)\end{aligned}$$

where

$$J(a,i)_s = \begin{cases} \beta(s)_l(i)(\beta(s)_l(i)^{-1}(f_s(\alpha_l(i)(a)))) & \text{if } x(s,i)=l\\ \beta(s)_l(i)((f_s((a)\alpha_r(i)))\beta(s)_r(i)^{-1}) & \text{if } x(s,i)=r\end{cases}$$

and

$$K(a,i)_s = \begin{cases} f_s(\alpha_l(i)(a)) & \text{if } x(s,i)=l\\ (\beta(s)_l(i)(f_s((a)\alpha_r(i))))\beta(s)_r(i)^{-1} & \text{if } x(s,i)=r\end{cases}$$

Since f_s is I-equivariant for all $s\in S$ we have

$$\begin{aligned}\text{LHS} &= (g((L(a,i)_s)_{s\in S}))\gamma_r(i)\\ &= (g((f_s(\alpha_l(i)(a)))_{s\in S})\gamma_r(i)\\ &= (g(f(\alpha_l(i)(a)))\gamma_r(i)\\ &= \text{RHS}\end{aligned}$$

where

$$L(a,i)_s = \begin{cases} f_s(\alpha_l(i)(a)) & \text{if } x(s,i)=l\\ ((f_s(\alpha_l(i)(a)))\beta(s)_r(i))\beta(s)_r(i)^{-1} & \text{if } x(s,i)=r\end{cases}$$

This completes the proof. $\square$

Proposition 3.2. *Let $f : A \to B$ be a bijective equivariant function where (A, α) and (B, β) are semi-invertible finite I-sets. Then the inverse f^{-1} is equivariant.*

Proof. Assume f is equivariant. We want to show

$$\alpha_l(i)(f^{-1}((b)\beta_r(i))) = (f^{-1}(\beta_l(i)(b)))\alpha_r(i).$$

Assume first both $\alpha_l(i)$ and $\beta_l(i)$ is isomorphism. First since both f and $\alpha_l(i)$ are bijective, we can write

$$\beta_l(i)((b)\beta_r(i)) = (f(\alpha_l(i)\alpha_l(i)^{-1}(f^{-1}(\beta_l(i)(b))))\beta_r(i).$$

Since f is equivariant then

$$\beta_l(i)((b)\beta_r(i)) = \beta_l(i)(f(\alpha_l(i)^{-1}((f^{-1}(\beta_l(i)(b)))\alpha_r(i)))).$$

and since $\beta_l(i)$ is bijective we get

$$(b)\beta_r(i) = f(\alpha_l(i)^{-1}((f^{-1}(\beta_l(i)(b)))\alpha_r(i)))$$

which implies

$$\alpha_l(i)(f^{-1}((b)\beta_r(i))) = (f^{-1}(\beta_l(i)(b)))\alpha_r(i).$$

The case when both $\alpha_r(i)$ and $\beta_r(i)$ is isomorphism is similar. Assume now $\alpha_r(i)$ and $\beta_l(i)$ are isomorphisms. Since f is an isomorphism, then the composition of f^{-1}, $\alpha_r(i)$ and $\beta_l(i)$ is an isomorphism. Since A and B are finite sets then from the equality

$$(f(\alpha_l(i)(a)))\beta_r(i) = \beta_l(i)(f((a)\alpha_r(i)))$$

we get $\alpha_l(i)$ and $\beta_r(i)$ are isomorphisms as well. Hence, f^{-1} is equivariant. The case $\alpha_r(i)$ and $\beta_l(i)$ are isomorphism is the same. Hence this proves the statement. $\square$

Observe that if the semi-invertible actions are isomorphism in same sides, then we do not need the finiteness assumption. However, in general this proposition is not correct when we drop the assumption on finiteness. For example if $I = \mathbb{N}$ and $A = B = \mathbb{N}$ with the actions $\alpha = (\alpha_l, 1)$ on A such that $\alpha_l(1)(i) = i + 1$ and $\beta = (1, \beta_r)$ on B such that $\beta_r(1)(i + 1) = i$ and $\beta_r(1)(0) = 0$, then the identity function $id : A \to B$ is equivariant but $id : B \to A$ is not.

3.3. **Categories of I-sets and equivalence of view points.** Let I be a semigroup, considering the usual definition of actions we let $\mathrm{ACT}_l(I)$, $\mathrm{ACT}_r(I)$, $\mathrm{act}_l(I)$, $\mathrm{act}_r(I)$ to denote the category of left I-sets, right I-sets, finite left I-sets, and finite right I-sets respectively and I-equivariant maps. Now we define four new categories denoted by $\mathrm{ACT}(I)$, $\mathrm{act}(I)$, $\overline{\mathrm{ACT}}(I)$ and $\overline{\mathrm{act}}(I)$. The objects of the categories $\mathrm{ACT}(I)$ and $\mathrm{act}(I)$ are I-sets which are products of semi-invertible I-sets and finite I-sets which are products of semi-invertible I-sets respectively, where I-sets are defined as in the previous section. The objects of $\overline{\mathrm{ACT}}(I)$ and $\overline{\mathrm{act}}(I)$ are I-sets which are products of sets with actions that are invertible on one side and finite I-sets which are products of sets with actions that are invertible on one side respectively, where again I-sets are defined as in the previous section. The morphisms of the categories $\mathrm{ACT}(I)$, $\mathrm{act}(I)$, $\overline{\mathrm{ACT}}(I)$, $\overline{\mathrm{act}}(I)$ are I-equivariant functions (defined as in Section 2.2). Now the following result shows that Definition 2.1 is equivalent to the usual one for groups.

Theorem 3.3. *For a group G, the categories* $\mathrm{act}(G)$*,* $\overline{\mathrm{act}}(G)$*,* $\mathrm{act}_l(G)$ *and* $\mathrm{act}_r(G)$ *are all equivalent to each other as categories and* $\mathrm{ACT}(G)$*,* $\overline{\mathrm{ACT}}(G)$*,* $\mathrm{ACT}_l(G)$*,* $\mathrm{ACT}_r(G)$ *are all equivalent to each other as categories.*

Proof. Here we will only prove the equivalence of $\mathrm{ACT}(G)$ and $\mathrm{ACT}_l(G)$ the rest is either similar or just obtained by restrictions of the equivalences. Define a functor

$$F : \mathrm{ACT}(G) \to \mathrm{ACT}_l(G)$$

which sends an object (A, α) in $\mathrm{ACT}(G)$ to the left action $\mu : G \to \mathrm{End}_l(A)$ given by

$$\mu(g)(a) = \alpha_l(g)((a)\alpha_r(g^{-1}))$$

and sends a morphism f from (A, α) to (B, β) to itself considered as a function from A to B. We also define a functor

$$H : \mathrm{ACT}_l(G) \to \mathrm{ACT}(G)$$

which sends $\mu : G \to \mathrm{End}_l(A)$ to $(A, (\mu, \iota_r))$ where ι_r is the constant function from G to $\mathrm{End}_r(A)$ sending every element of G to identity function on A and sends a morphism to itself considered as a function again. Now clearly $F \circ H$ is identity on $\mathrm{ACT}_l(G)$. We can define a natural transformation from $H \circ F$ to identity on $ACT(G)$ by sending each object (A, α) in $\mathrm{ACT}(G)$ to the G-equivariant function from (A, α) to $(H \circ F)(A, \alpha)$ given by identity function on A. Now it is straight forward to check that this gives an equivalence between $\mathrm{ACT}(G)$ and $\mathrm{ACT}_l(G)$. $\square$

4. Action inverting functors

For a semigroup I we define four semigroup homomorphisms as follows: The homomorphisms

$$\iota_l : I \to \mathrm{End}_l(I) \quad \text{and} \quad \iota_r : I \to \mathrm{End}_r(I)$$

sends every element to identity endofunction and the homomorphisms

$$\mu_l : I \to \mathrm{End}_l(I) \quad \text{and} \quad \mu_r : I \to \mathrm{End}_r(I)$$

are given by multiplication from left and right respectively.

4.1. **Inverting actions from left to right.** Consider I itself as a I-set with the action (ι_l, μ_r). Let A be a set with a I-action α. To indicate the right action on I is trivial let us denote the set of equivariant functions, $\mathrm{Map}_I(I, A)$, by $\mathrm{Inv}^{\mathrm{r}}_{\mathrm{I}}(A)$. Let $f : I \to A$ be a I-equivariant map, i.e., for every i, j in I we have

$$(f(j))\alpha_r(i) = \alpha_l(i)(f(j \otimes i))$$

We define a I-action $\theta = (\theta_l, \theta_r)$ on $\mathrm{Inv}^{\mathrm{r}}_{\mathrm{I}}(A)$ as follows: The left component

$$\theta_l : I \to \mathrm{End}_l(\mathrm{Inv}^{\mathrm{r}}_{\mathrm{I}}(A))$$

sends an element k in I to the function

$$\theta_l(k) : \mathrm{Inv}^{\mathrm{r}}_{\mathrm{I}}(A) \to \mathrm{Inv}^{\mathrm{r}}_{\mathrm{I}}(A)$$

defined as the identity function. Hence the function $\theta_l(k)$ sends f to f. The right component

$$\theta_r : I \to \mathrm{End}_r(\mathrm{Inv}^{\mathrm{r}}_{\mathrm{I}}(A))$$

sends an element k in I to the function

$$\theta_r(k) : \mathrm{Inv}^{\mathrm{r}}_{\mathrm{I}}(A) \to \mathrm{Inv}^{\mathrm{r}}_{\mathrm{I}}(A)$$

defined as the function that sends f to the composition

$$I \xrightarrow{\mu_l(k)} I \xrightarrow{f} A$$

so that we have $(f)\theta_r(k)(j) = f(k \otimes j)$, for every $j, k \in I$. Since I is semi-invertible, then θ is well defined, by Theorem 3.1.

We call this action the inverse (from left to right) action of α. In fact this construction is functorial on $\mathrm{act}(I)$ and we denote the functor sending an I-action on a set A to the inverse I-action on $\mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}(A)$ by

$$\mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}} : \mathrm{act}(I) \to \mathrm{act}(I).$$

This functor sends a morphism $f : A \to B$ to the morphism which sends $h : I \to A$ to the composition $f \circ h$ from I to B. Given I-set A we define the evaluation function

$$\mathcal{E}_{\mathrm{l}}^{\mathrm{r}} : \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}(A) \to A$$

given by $\mathcal{E}_{\mathrm{l}}^{\mathrm{r}}(f) = f(1)$.

Lemma 4.1. *$\mathcal{E}_{\mathrm{l}}^{\mathrm{r}}$ defines a natural transformation from $\mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}$ to id, the identity functor.*

Proof. Let A be an I-set with action α. Since

$$\alpha_l(i)(\mathcal{E}_{\mathrm{l}}^{\mathrm{r}}((f)\theta_r(i))) = \alpha_l(i)(f(i)) = (f(1))\alpha_r(i) = (\mathcal{E}_{\mathrm{l}}^{\mathrm{r}}(f))\alpha_r(i)$$

then $\mathcal{E}_{\mathrm{l}}^{\mathrm{r}}$ is equivariant, so that it defines a natural transformation from $\mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}$ to *id*. □

Let G be a group. We define a functor

$$\mathrm{inv}_l^r : \mathrm{act}_l(G) \to \mathrm{act}_r(G)$$

which sends a left G action

$$\nu : G \times X \to X, \text{ given by } (g, x) \mapsto g.x$$

for $g \in G$ and $x \in X$, to a right G-action

$$\nu^{-1} : X \times G \to X, \text{ given by } (x, g) \mapsto g^{-1}x$$

for $g \in G$ and $x \in X$, which is called the inverse action of ν. The following theorem shows that the two definitions we gave for inverse actions agree for actions of groups.

Theorem 4.2. *The diagram*

$$\begin{tikzcd}
\mathrm{ACT}_l(G) \arrow[r, "\mathrm{inv}_l^r"] \arrow[d, "\cong"'] & \mathrm{ACT}_r(G) \arrow[d, "\cong"] \\
\mathrm{ACT}(G) \arrow[r, "\mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}"] & \mathrm{ACT}(G)
\end{tikzcd}$$

is commutative up to a natural isomorphism.

Proof. Given $\mu : G \to \mathrm{End}_l(X)$ a left G-action on X. The inverse action of G on $\mathrm{Map}_G(G, X)$ is obtained by considering $(\mu, 1)$ as a G-action on X and G itself as a G-set with the action (ι_l, μ_r). The evaluation function $\mathcal{E}_{\mathrm{l}}^{\mathrm{r}}$ from $\mathrm{Map}_G(G, X)$ to the G-set X is a G-equivariant bijection. It is also straight forward to check that the assignment that sends (X, μ) to ${\mathcal{E}_{\mathrm{l}}^{\mathrm{r}}}_{(X,\mu)}$ gives a natural isomorphism which shows that the diagram in the theorem is commutative up to natural isomorphism. □

4.2. **Inverting actions from right to left.** We can also invert actions from right to left. This time we consider I as an I-set with the action (μ_l, ι_r), so that an I-equivariant function $f : I \to A$ satisfies

$$(f(i \otimes j))\alpha_r(i) = \alpha_l(i)(f(j))$$

for every i, j in I. In this case we denote the set of equivariant functions from I to A, $\mathrm{Map}_I(I, A)$, by $\mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A)$. We define a I-action $\vartheta = (\vartheta_l, \vartheta_r)$ on $\mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A)$ as follows: The left component

$$\vartheta_l : I \to \mathrm{End}_l(\mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A))$$

sends an element k in I to the function

$$\vartheta_l(k) : \mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A) \to \mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A)$$

defined as the function that sends f to the composition

$$I \xrightarrow{\mu_r(k)} I \xrightarrow{f} A$$

so that we have $\vartheta_l(k)(f)(i) = f(i \otimes k)$. The right component

$$\vartheta_r : I \to \mathrm{End}_r(\mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A))$$

sends an element k in I to the function

$$\vartheta_r(k) : \mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A) \to \mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A)$$

defined as the identity function. Hence the function $\vartheta_r(k)$ sends f to f. Again by Theorem 3.1 this construction is well defined. Similar to the Lemma 4.1, there is an equivariant evaluation function

$$\mathcal{E}_{\mathrm{r}}^{\mathrm{l}} : \mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A) \to A$$

given by $\mathcal{E}_{\mathrm{r}}^{\mathrm{l}}(f) = f(1)$, which is equivariant.

A version of Theorem 4.2 is also true for this case, i.e. the diagram

$$\begin{tikzcd} \mathrm{ACT}_r(G) \arrow[r, "\mathrm{inv}_r^l"] \arrow[d, "\cong"'] & \mathrm{ACT}_l(G) \arrow[d, "\cong"] \\ \mathrm{ACT}(G) \arrow[r, "\mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}"] & \mathrm{ACT}(G) \end{tikzcd}$$

is commutative up to a natural isomorphism, where G is a group and the horizontal functors defined analogues to the ones in the Section 4.1. The proof is the same as proof of Theorem 4.2. When G is a group the composition $\mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}} \circ \mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}$ is identity functor.

4.3. **As idemponent endofunctors on $\overline{\mathrm{ACT}}(I)$.** Let I be a monoid. The following lemma shows that the inverting functors are idempotent.

Proposition 4.3. *The evaluations function $\mathcal{E}_{\mathrm{l}}^{\mathrm{r}}$ (resp. $\mathcal{E}_{\mathrm{r}}^{\mathrm{l}}$) defines a natural isomorphisms from $\mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}} \circ \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}$ to $\mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}$ (resp. from $\mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}} \circ \mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}$ to $\mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}$).*

Proof. For any I-set A, consider the function

$$\Phi_A : \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}(A) \to \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}} \circ \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}(A)$$

given by

$$\Phi(g)(i)(j) = g(i \otimes j)$$

for $g \in \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}(A)$ and $i, j \in I$. It is straightforward to check the equalities

$$\alpha_l(k)(\Phi(g)(i)(j \otimes k)) = (\Phi_A(g)(i)(j))\alpha_l(k)$$

and

$$\Phi(g)(i \otimes k) = (\Phi(g)(i))\theta_r(k)$$

so that ϕ is well defined. Since

$$g(k \otimes i \otimes j) = \Phi((g)\theta_r(k))(i)(j) = (\Phi(g))\theta_r(k)(i)(j) = g(k \otimes i \otimes j)$$

then Φ is equivariant. For any $g \in \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}(A)$ we have

$$(\mathcal{E}_{\mathrm{l}}^{\mathrm{r}} \circ \Phi)(g)(i) = \Phi(g)(i)(1) = g(i)$$

and for any $h \in \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}} \circ \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}}(A)$ we have

$$(\Phi \circ \mathcal{E}_{\mathrm{l}}^{\mathrm{r}})(h)(i)(j) = \Phi(h(1))(i)(j) = h(1)(i \otimes j) = h(i)(j)$$

so that $\mathcal{E}_{\mathrm{l}}^{\mathrm{r}}$ and Φ are mutual inverses. This completes the proof. The same proof works for $\mathcal{E}_{\mathrm{r}}^{\mathrm{l}}$ as well. □

We denote the composition of two inverse endo-functors on $\overline{\mathrm{ACT}}(I)$ by INV, in other words we have

$$\mathrm{INV} = \mathrm{Inv}_{\mathrm{l}}^{\mathrm{r}} \circ \mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}$$

considered as an endofunctor on $\overline{\mathrm{ACT}}(I)$. Then for any I-set M we have an I-equivariant function

$$\mathcal{E} : \mathrm{INV}(M) \to M$$

defined by $\mathcal{E}(f) = f(1)(1)$. Then $\mathcal{E}$ defines a natural transformation from $\mathrm{INV} \circ \mathrm{INV}$ to INV. When I is a commutative monoid, then it is an isomorphism.

Proposition 4.4. *If I is a commutative monoid then $\mathcal{E}$ defines a natural isomorphism from* $\mathrm{INV} \circ \mathrm{INV}$ *to* INV.

Proof. For any I-set X, the function

$$\Phi_X : \mathrm{INV}(X) \to \mathrm{INV} \circ \mathrm{INV}(X)$$

given by

$$\Phi_X(g)(i)(j)(k)(l) = g(i \otimes k)(j \otimes l)$$

for $g \in \mathrm{INV}(X)$ and $i, j, k, l \in I$. It is straightforward to check that this function is equivariant since on both $\mathrm{INV}(X)$ and $\mathrm{INV} \circ \mathrm{INV}(X)$ the right actions are trivial. We have

$$\mathcal{E}(X)(\Phi_X(g))(k)(l) = g(k)(l)$$

and

$$\Phi_X(\mathcal{E}(X)(g))(k)(l) = g(k)(l)$$

so that $\mathcal{E}(X)$ and Φ_X are mutual inverses. This completes the proof. □

4.4. **Inverse actions on finite sets.** We again use the same notations for the restrictions of $\mathrm{Inv}^{\mathrm{r}}_{1}$, $\mathrm{Inv}^{\mathrm{l}}_{\mathrm{r}}$ and their compositions INV on $\overline{\mathrm{act}}(I)$. Let (A,α) be an I-set such that the right action α_r is trivial. For an element x in A let Ix denote the orbit set $Ix = \{\alpha_l(i)(x) : i \in I\}$ and $If(I)$ denote the set $If(I) = \{\alpha_l(i)(x) : i \in I, x \in im(f)\}$. We define a set A^l as

$$\mathrm{A}^l = \{x \in A : \ \textit{for all } i \in I, \ \alpha_l(i)|_{Ix} \textit{ is one-to-one}\}$$

which is invariant under the action $\alpha = (\alpha_l, 1)$.

Proposition 4.5. *Let I be a monoid and let A be a finite set. Let (A,α) be an I-set such that the right action α_r is trivial. Then there is a bijection $\mathrm{Inv}^{\mathrm{r}}_{1}(A) \cong \mathrm{A}^l$.*

Proof. Firstly, for an element $x \in \mathrm{A}^l$ we define $f_x : I \to A$ with $f_x(i) = \alpha_l(i)^{-1}(x)$, then since $x \in \mathrm{A}^l$ this is a well-defined map. By definition for every i, j in I we have

$$\alpha_l(i)f_x(j \otimes i) = \alpha_l(i)\alpha_l(j \otimes i)^{-1}(x) = \alpha_l(j)^{-1}(x) = f_x(j).$$

Hence f_x is equivariant and we have an injective function $\mathrm{A}^l \to \mathrm{Inv}^{\mathrm{r}}_{1}(A)$.

Now suppose that $f : I \to A$ be a function in $\mathrm{Inv}^{\mathrm{r}}_{1}(A)$. We claim that $f(1)$ is an element of A^l. Assume the contrary that there exist i, j, k in I such that

$$\alpha_l(j)(f(1)) \neq \alpha_l(k)(f(1)) \ \text{ and } \ \alpha_l(i \otimes j)(f(1)) = \alpha_l(i \otimes k)(f(1)).$$

Since A is finite then for every $i \in I$ there exist positive integers m, m' with $m > m'$ such that for all x in $If(I)$, we have the identity $\alpha_l(i^m)(x) = \alpha_l(i^{m'})(x)$. Hence the restriction of $\alpha_l(i^{m-m'})$ to the set

$$\alpha_l(i^{m'})(If(I)) := \{\alpha_l(i^{m'})(x) : x \in If(I)\}$$

is the identity function. Moreover, for any $v \in I$ we have

$$f(v) = \alpha_l(i^{m'})f(v \otimes i^{m'})$$

so that $im(f)$ is contained in $\alpha_l(i^{m'})(If(I))$.

Let j and k be two elements in I. As above there are integers t, t' with $t > t'$ and

$$\alpha_l(j^{t'})f(j^t) = \alpha_l(j^{t'})f(j^{t'})$$

so that $\alpha_l(j)(f(1)) = f(j^{t-t'-1})$. Similarly there are integers s, s' with $s > s'$ and $\alpha_l(k)(f(1)) = f(k^{s-s'-1})$. Hence both $\alpha_l(k)(f(1))$ and $\alpha_l(k)(f(1))$ are elements of $im(f)$, which means $\alpha_l(i^{m-m'})$ is identity on both.

By our initial assumption we have

$$\alpha_l(i^{m-m'-1})(\alpha_l(i \otimes j)(f(1))) = \alpha_l(i^{m-m'-1})(\alpha_l(i \otimes k)(f(1)))$$

which implies

$$\alpha_l(i^{m-m'})(\alpha_l(j)(f(1))) = \alpha_l(i^{m-m'})(\alpha_l(k)(f(1)))$$

As a result we get $\alpha_l(j)(f(1)) = \alpha_l(k)(f(1))$, i.e. a contradiction, so that $f(1)$ must be an element of A^l. The association $f \mapsto f(1)$ defines a function and by definition of A^l this function is injective as well. Thus, we get a bijection as desired. This completes the proof. $\square$

Since in $\overline{\text{act}}(I)$ one side is an isomorphism, then this lemma implies the inverse of an object in $\overline{\text{act}}(I)$ is also finite, in fact there is a one to one correspondence with the elements in A^l.

In the case when the left action α_l is trivial one can define the orbit set $xI = \{(x)\alpha_r(i) : i \in I\}$ and the set $f(I)I = \{(x)\alpha_r(i) : i \in I, x \in im(f)\}$. We can define a set

$$\mathrm{A}^r = \{x \in A : \ for\ all\ i \in I,\ \alpha_r(i)|_{xI}\ is\ one\text{-}to\text{-}one\}$$

so that A^r is invariant under the action $\alpha = (1, \alpha_r)$. In this case we have a similar lemma as follows:

Proposition 4.6. *There is a bijection* $\mathrm{Inv}_{\mathrm{r}}^{\mathrm{l}}(A) \cong \mathrm{A}^r$.

If $\overline{\mathcal{E}}$ is the restriction of $\mathcal{E}$ on finite I-sets. Note that $\overline{\mathcal{E}}$ is bijective by the previous propositions. We have the following lemma:

Proposition 4.7. $\overline{\mathcal{E}}$ *defines a natural isomorphism from* $\mathrm{INV} \circ \mathrm{INV}$ *to* INV.

Proof. This directly follows from Proposition 3.2, since $\overline{\mathcal{E}}$ is bijective. □

5. Homotopical category structure on $\overline{\mathrm{ACT}}(I)$

In this section we discuss homotopical category structure on $\overline{\mathrm{ACT}}(I)$ where I is a monoid. We refer [7] for general terminology and homotopical notions in this section. Let A, B be I-sets in $\overline{\mathrm{ACT}}(I)$ and $f : A \to B$ be an I-equivariant map. We say f is a weak equivalence if the induced function $\mathrm{INV}(f) : \mathrm{INV}(A) \to \mathrm{INV}(B)$ is an isomorphism. We denote the class of weak equivalences by W. It is straightforward to check that these weak equivalences satisfy 2-out-of-6 property. Hence this makes $\overline{\mathrm{ACT}}(I)$ into a homotopical category. We denote its homotopy category by $\mathrm{Ho}(\overline{\mathrm{ACT}}(I))$.

5.1. **Saturation of the category of finite I-sets.** For details of 3-arrow calculus we refer [7], 27.3. Notice that $\overline{\text{act}}(I)$ is a homotopical category. We will show that $\overline{\text{act}}(I)$ admits a 3-arrow calculus. To do this we define two subclasses U and V of the class weak equivalences W of $\overline{\text{act}}(I)$ as follows: U will be the subclass of W which are also inclusions and V will be the subclass of W which are also surjections. Assume now we have a zig-zag $A' \xleftarrow{u} A \xrightarrow{f} B$ in $\overline{\text{act}}(I)$ where u is in U. Then we can associate another zig-zag $A' \xrightarrow{f'} B' \xleftarrow{u'} B$ from the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "u'"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

so that $f' \circ u = u' \circ f$. Hence that the function u' is an inclusion and weak equivalence, i.e. u' is in U. The functions u' and f' are induced by inclusions so that they are equivariant. If u is an isomorphism then u' is also an isomorphism since both u and u' fits in above pushout diagram. Similarly if we have a zig-zag $X \xrightarrow{g} Y \xleftarrow{v} Y'$ in $\overline{\text{act}}(I)$ then we can associate another zig-zag $X \xleftarrow{v'} X' \xrightarrow{g'} Y$ from

the pullback diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "v'"'] & Y' \arrow[d, "v"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

so that $g \circ v' = v \circ g'$, and again if u is an isomorphism then so does u'. Assume $w : M \to N$ is a weak equivalence in $\overline{\text{act}}(I)$, then consider the pushout diagram

$$\begin{tikzcd} \text{INV}(M) \arrow[r, "\mathcal{E}"] \arrow[d, "w \circ \mathcal{E}"'] & M \arrow[d, "u"] \\ N \arrow[r, "\tilde{u}"] & M' \end{tikzcd}$$

An equivariant function f in $\text{INV}(M)$ satisfies

$$f(i \otimes j)(i \otimes k) = f(j)(k)$$

for every i, j and k in I. Now assume $\mathcal{E}(f) = \mathcal{E}(g)$, i.e. $f(1)(1) = g(1)(1)$. Then since both $f(1)$ and $g(1)$ are equivariant in $\text{Inv}^{\text{r}}_{\text{l}}(M)$, then

$$(f(1)(j))\alpha_r(k) = \alpha_l(k)(f(1)(k \otimes j)) \text{ and } (g(1)(j))\alpha_r(k) = \alpha_l(k)(g(1)(k \otimes j)).$$

Since (A, α) is in $\overline{\text{act}}(I)$ then one of $\alpha_l(k)$ or $\alpha_r(k)$ is invertible. If $\alpha_l(k)$ is invertible then we have

$$\alpha_l(k)^{-1}(f(1)(1))\alpha_r(k) = (f(1)(k)) \text{ and } \alpha_l(k)^{-1}(g(1)(j))\alpha_r(k) = g(1)(k),$$

so that $f(1) = g(1)$ and as in proof of 4.5 $f = g$, i.e. $\mathcal{E}$ is injective. If γ is the inverse of the inverse action on A, i.e. action on $\text{Inv}^{\text{l}}_{\text{r}}(\text{Inv}^{\text{r}}_{\text{l}}(M))$, then we have

$$(\mathcal{E}(\gamma_l(i)(f)))\alpha_r(i) = (f \circ \mu_r(i)(1)(1))\alpha_r(i) = (f(i)(1))\alpha_r(i)$$

by equivariance of $f(i)$ this is equal to

$$\alpha_l(i)(f(i)(i)) = \alpha_l(i)(f(1)(1)) = \alpha_l(i)(\mathcal{E}(f))$$

hence, $\mathcal{E}$ is equivariant.

Since the above square is a pushout then $\tilde{u}$ is injective. Hence, there is a unique function $v : M' \to N$ which is surjective. As before, the functions u and v are also equivariant, so that we have a factorization of w as $w = v \circ u$ such that v is in V and u is in U. Hence $\overline{\text{act}}(I)$ admits a 3-arrow calculus. Let us denote the homotopy category of $\overline{\text{act}}(I)$ by $\text{Ho}(\overline{\text{act}}(I))$ and let $L : \overline{\text{act}}(I) \to \text{Ho}(\overline{\text{act}}(I))$ be the localization with respect to the above weak equivalences (see [7] 26.5). Then by 27.5 of [7] we can conclude that $\overline{\text{act}}(I)$ is saturated, i.e. a function in $\overline{\text{act}}(I)$ is a weak equivalence if and only if its image in $\text{Ho}(\overline{\text{act}}(I))$, under the localization functor, is an isomorphism.

Note that it is possible to define stronger classes of weak equivalences on these categories which still make them homotopical categories, by using similar ideas above along with restrictions of actions to submonoids or subsets. However, not all of them admit a 3-arrow calculus. For a given a submonoid J of I let Res^I_J : $\overline{\text{act}}(I) \to \overline{\text{act}}(J)$ be the restriction functor, which sends a finite I-set (A, α) to A with the restriction of α on J. Let Y be a collection of submonoids of I which contains I. A function $f : A \to B$ in $\overline{\text{act}}(I)$ is called a Y-equivalence if for every

J in Y the function $Res^I_J(f)^* : \mathrm{Map}_J(J, A) \to \mathrm{Map}_J(J, A)$ is an I-equivariant isomorphism. Since Res^I_J respects compositions, then the class of Y-equivalences satisfy both 2-out-of-3 and 2-out-of-6 property and so that again $\overline{\mathrm{act}}(I)$ with Y-equivalences will be a homotopical category admitting a 3-arrow calculus, when we set U as the subclass of Y-equivalences which are inclusions and V as the subclass of Y-equivalences which are surjections. It is now straightforward to check that these classes satisfied the required axioms. A Y-equivalence is trivially a weak equivalence so that Y-equivalences are stronger form of weak equivalences. In this paper we continue with the weak equivalences instead of Y-equivalences for convenience.

6. Burnside ring

In the classical theory of group actions, when a group G is given, the Burnside ring of G, denoted by $A(G)$, is defined as the Grø"thendieck ring of the semiring of isomorphism classes of finite G-sets. The Burnside ring of a group is a very important construction in the group theory, and has several applications, see e.g. [11], [5], [4], [6]. We define the Burnside ring of a monoid by using saturation of the homotopical structure of $\overline{\mathrm{act}}(I)$. The isomorphism classes in $\mathrm{Ho}(\overline{\mathrm{act}}(I))$ forms a semiring under disjoint union as addition and cartesian product as multiplication. We call the Gröthendieck ring of this semiring as the Burnside ring of I, and we denote this ring by $\mathrm{A}(I)$. Let us denote by $K(I)$ the Gröthendieck group of a monoid I. Then $\mathrm{A}(K(I))$ denotes the usual Burnside ring of the group $K(I)$ (see e.g. [11]). Most of the properties of this Burnside ring follows from the Section 4.4.

By definition the Burnside ring of a group given in this way is equal to the standard one. The following proposition shows that the definitions of Burnside ring of a commutative monoid is same as Burnside ring of its Gröthendieck construction. Hence, it does validate the name Burnside ring of a monoid.

Theorem 6.1. *For any commutative monoid I we have $\mathrm{A}(I)$ is isomorphic to $\mathrm{A}(K(I))$.*

Proof. There is an obvious function $\Lambda : \mathrm{A}(K(I)) \to \mathrm{A}(I)$. Hence we need to define the inverse. Let M be an I-set, then by Proposition 4.3, the I-equivariant function $\mathcal{E} : \mathrm{INV}(M) \to M$ is a weak equivalence and on $\mathrm{INV}(M)$ we have an action of $K(I)$, since both components of $\mathrm{INV}(M)$ are bijective. Hence, any finite I-set defines a finite $K(I)$-set. Define a function $\Gamma : \mathrm{A}(I) \to \mathrm{A}(K(I))$ by sending a class $[M]$ of I-set M in $\mathrm{A}(I)$ to the class $[\mathrm{INV}(M)]$. Since $\mathcal{E}$ induces isomorphism on the inverse actions then $[M] = [\mathrm{INV}(M)]$ in $\mathrm{A}(I)$. Hence Γ is a ring isomorphism with the inverse Λ. □

Note that one can also define Burnside ring with Y-equivalences on $\overline{\mathrm{act}}(I)$ defined in the previous section, which again will coincide with the definition of Burnside ring of a group. However, in this case for an arbitrary monoid the Burnside ring is much bigger and has a very complicated structure, so that the classification problems would become very difficult.

6.1. **Attractors of a state machine.** It is well known that the notion of attractors and attracting sets play an important role in non-linear physics, Geometry and in particular in the theory of dynamical systems, see for example [1], [9], [2]. We define analogues notion for state machines. We will consider semiautomaton in the following sense. Let I be a free semigroup on an alphabet and X be an I-set with

action $(\alpha_l, 1)$, i.e. the right action is trivial. Assume both I and X has a topology and the action is continuous.

Definition 6.2. An attractor A of this action is a subset of X defined by the following properties: There exists neighborhood of X in X denoted by $B(X)$ called a basin of attraction for A such that for all submonoid I' of I we have

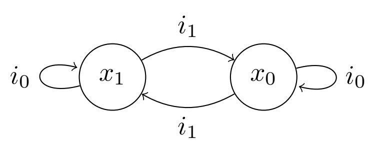
$$A = \coprod_{s \in S} A_s$$

for some index set S so that the following holds for every s in S:

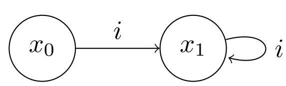
(1) A_s is forward invariant, i.e. $I'A_s \subset A_s$.
(2) For every b in $B(A)$ there is a word w such that $\alpha_l(w)(b)$ is in A_s.
(3) A_s is a minimal set satisfying the above two properties.

This definition can be viewed as a special case of the definition given in [10], when we consider attractors as global uniform attractors in the basin of attraction. In this paper we only use discrete topology on sets. We will say that an attractor is periodic if it is invariant up to isomorphism under the endofunctor INV. Notice that the Burnside ring of I is generated by periodic attractors for finite I-sets with discrete topology. Hence, the Burnside ring can be used to understand types of periodic attractors which is also important for the usual definition of periodic attractors.

As a simple example one can consider the set with two elements $\{x_0, x_1\}$ with the action of free monoid with two generators $\{i_0, i_1\}$ as follows

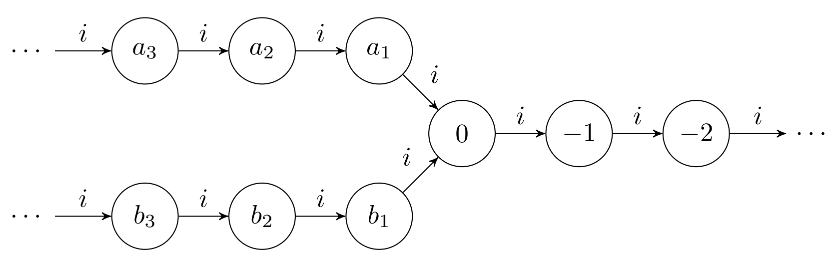


In this case the inverse action will be isomorphic to itself, and the attractor is the all of the set. If we consider the action on $\{x_0, x_1\}$ with the action of free monoid with one generator i, given by



then the inverse action will be singleton with trivial action on it, so the attractor in this case is $\{x_1\}$.

As a last remark we should note that Proposition 4.5 is not valid in the case when A is infinite. For example let $I \cong \mathbb{N}$ with a single generator i and the set A and the action of I be as in the figure below



Then the inverse action will be as follows:

$$\begin{tikzcd}
\cdots & f_{b_3} \arrow[l, "i"'] & f_{b_2} \arrow[l, "i"'] & f_{b_1} \arrow[l, "i"'] & f_{a_0} \arrow[l, "i"'] & f_{b_{-1}} \arrow[l, "i"'] & f_{b_{-2}} \arrow[l, "i"'] & \cdots \arrow[l, "i"']
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots & f_{a_3} \arrow[l, "i"'] & f_{a_2} \arrow[l, "i"'] & f_{a_1} \arrow[l, "i"'] & f_{b_0} \arrow[l, "i"'] & f_{a_{-1}} \arrow[l, "i"'] & f_{a_{-2}} \arrow[l, "i"'] & \cdots \arrow[l, "i"']
\end{tikzcd}$$

where f_{a_k} and f_{b_k} are defined by $f_{a_k}(i^n) = a_{n+k}$ and $f_{b_k}(i^n) = b_{n+k}$ for $n, k \in \mathbb{N}$.

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